Multiple phase transitions in long range first-passage percolation on square lattice

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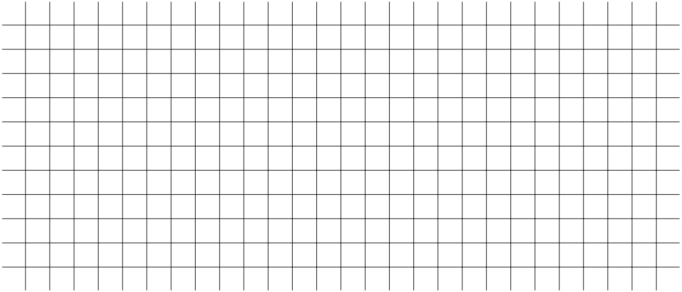
Joint with Partha Dey

Eindhoven Stochastics Seminar

Nearest Neighbor First passage percolation model on $\mathbb{Z}^d$



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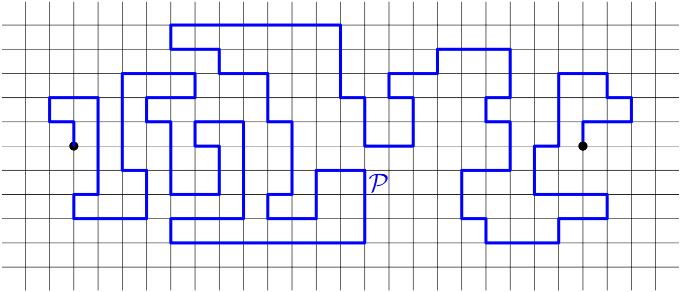

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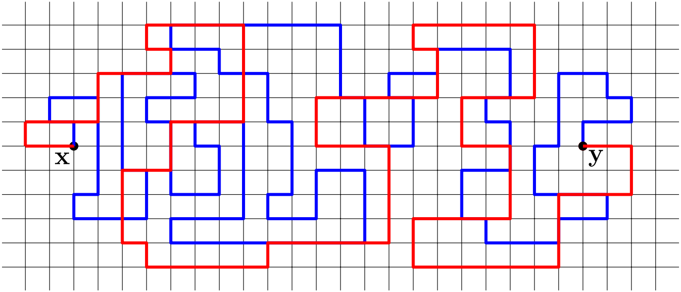

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- For a path $\mathfrak{P}$, the passage time for $\mathfrak{P}$ is defined as the sum of weights over all the edges in $\mathfrak{P}$.
- For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, the first-passage time $a(\mathbf{x}, \mathbf{y})$ is defined as the minimum passage time over all paths from $\mathbf{x}$ to $\mathbf{y}$.

Mean behavior

- The model was introduced by Hammersley and Welsh in 1965, where they proved that for all $\mathbf{x} \in \mathbb{Z}^d$

$$\nu(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(a(\mathbf{0}, n\mathbf{x}))$$

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- The shape theorem by Cox and Durrett('81) says that

$$\frac{1}{t}\{\mathbf{x} \in \mathbb{Z}^d : a(\mathbf{0}, \mathbf{x}) \leq t\} \oplus \left[-\frac{1}{2}, \frac{1}{2}\right]^d \xrightarrow{t\to\infty} B,$$

  where $B = \{\mathbf{x} : \nu(\mathbf{x}) \leq 1\}$ is a convex subset of $\mathbb{R}^d$.

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  where $B = \{\mathbf{x} : \nu(\mathbf{x}) \leq 1\}$ is a convex subset of $\mathbb{R}^d$.

- $\mathbb{V}\mathrm{ar}(a(\mathbf{0}, n\mathbf{x})) = o(n)$.

Shape Theorem

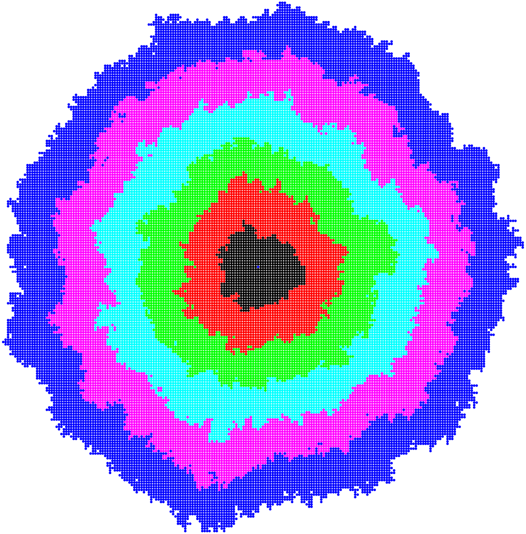

Figure: Limiting shape for $\{\mathbf{x} \in \mathbb{Z}^d : a(\mathbf{0}, \mathbf{x}) \leq t\}$.

Long-range First-Passage Percolation (LRFPP) on $\mathbb{Z}^d$

- Let $||\cdot||$ be the ℓ_1 norm on $\mathbb{Z}^d$.

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- Let $||\cdot||$ be the ℓ_1 norm on $\mathbb{Z}^d$.
- Now consider the complete graph on $\mathbb{Z}^d$ with unoriented edge set $\mathscr{E} = \{\langle \mathbf{x}\mathbf{y}\rangle : \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \mathbf{x} \neq \mathbf{y}\}$.
- Let $\{W_e : e \in \mathscr{E}\}$ be a collection of i.i.d. mean one exponentially distributed random variables.
- For a self avoiding path $\mathfrak{p} = \langle \mathbf{x}_0\mathbf{x}_1 \ldots \mathbf{x}_k\rangle$ with k edges, define its α-th passage time $W_{\mathfrak{p}}^{\alpha} := \sum_{i=1}^{k} ||\mathbf{x}_i - \mathbf{x}_{i-1}||^{\alpha} W_{\langle \mathbf{x}_{i-1}\mathbf{x}_i\rangle}$.
- For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, the α-th first-passage time is

$$T^{\alpha}(\mathbf{x}, \mathbf{y}) := \inf_{\mathfrak{p} \in \mathcal{P}_{\mathbf{xy}}} W_{\mathfrak{p}}^{\alpha},$$

where $\mathcal{P}_{\mathbf{xy}}$ is the set of all self-avoiding paths joining $\mathbf{x}$ and $\mathbf{y}$.

Question: How does $T^{\alpha}(\mathbf{0}, \mathbf{x})$ behave as $||\mathbf{x}||$ grows?

Alternative formulation for $\alpha > d$

$\alpha > d$ implies $\sum_{\mathbf{x} \in \mathbb{Z}^d} ||\mathbf{x}||^{-\alpha} < \infty$.

- This formulation is an extension of Richardson's model.
- Each site of $\mathbb{Z}^d$ is either occupied or vacant.
- Initially the origin is occupied only.
- Once $\mathbf{x}$ is occupied, it attempts to communicate at rate 1 and in each attempt it chooses a site $\mathbf{y}$ with probability $c||\mathbf{x} - \mathbf{y}||^{-\alpha}$ and makes it occupied.
- Occupied sites stay occupied.

Question: If $\mathcal{B}_t^\alpha$ is the set of vertices occupied by time t, then how does it grow?

Growth dynamics cartoon

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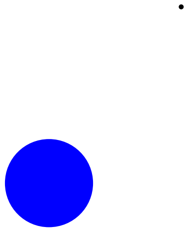

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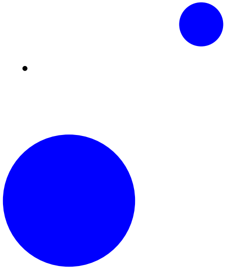

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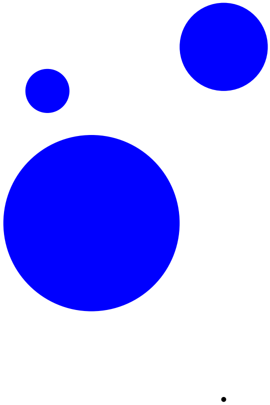

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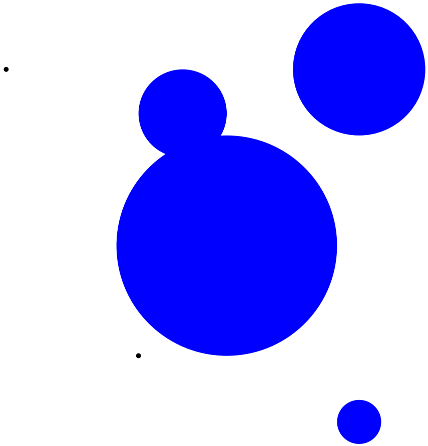

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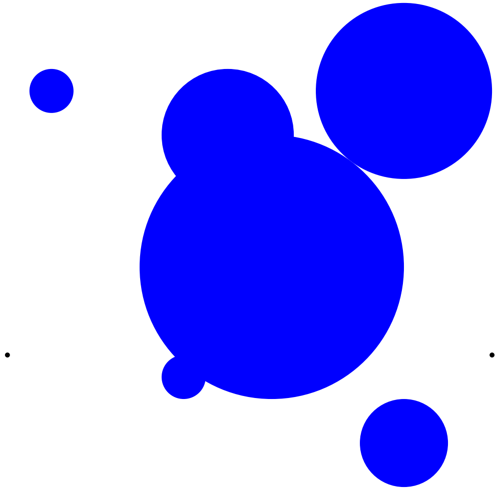

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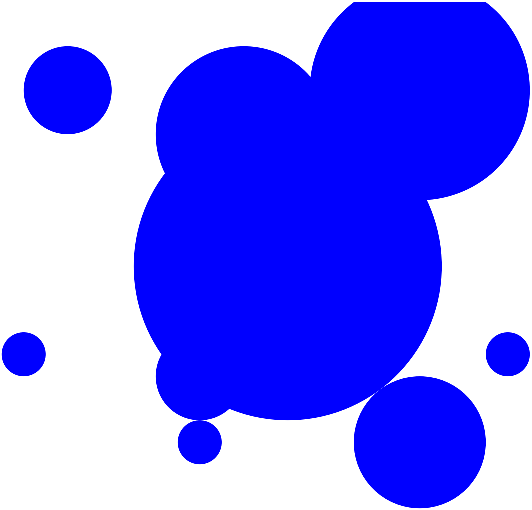

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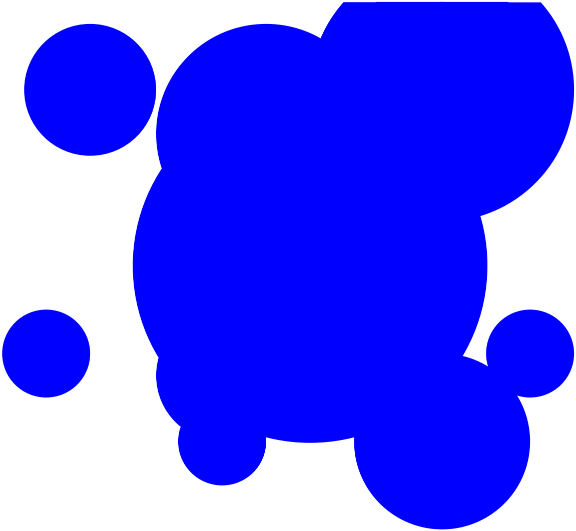

Appearance of LRFPP

- Mollison (1972) has considered similar models in the context of spatial propagation of epidemics. He proves linear growth in $d = 1$ for $\alpha > 3$.
- Cannas, Marco and Montemurro (2006) have considered long distance dispersal models in the context of biological invasion.
- Aldous (2007) has considered similar models on a torus in the context of random percolation of information through agent networks, and studied various related game theoretic aspects.
- C. and Durrett (2011) have considered a related continuous model (short-long FPP) on torus.
- Barbour and Reinert (arXiv) have considered gossip models on smooth Riemannian manifold.

Long-range percolation (LRP) results

$\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ are connected by an edge independently with prob. $p_{\mathbf{xy}} = |\mathbf{x} - \mathbf{y}|^{-\alpha + o(1)}$ as $|\mathbf{x} - \mathbf{y}| \to \infty$.
Let $D(.,.)$ be the associated random metric, $\mathcal{B}(\mathbf{x}, k)$ be the balls and D_L be the diameter of connected component of $[-L, L]^d$.

- LRP in one dimension:
  - Schulman ('83):
  - Aizenman and Newman, Newman and Schulman ('86)
  - Imbrie and Newman ('88)
  - Benjamini and Berger ('01):
- Coppersmith, Gamarnik and Sviridenko ('02): Diameter of LRP clusters.
- Biskup (2004, arXiv): Behavior of graph distance and D_L for $d < \alpha < 2d$.
- Trapman (2010): limits of $|\mathcal{B}(\mathbf{0}, k)|^{1/k}$ as $k \to \infty$.
- Berger (arXiv): Lower bound for $D(.,.)$ when $\alpha > 2d$

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Let D_L be the diameter of connected component of $[-L, L]^d$ in the associated random metric.

$$D_L \begin{cases} \to \lceil \alpha/(d-\alpha) \rceil, \alpha < d & \text{(Cor. of Benjamini, Kesten Peres and Schramm ('04))} \\ \asymp \log L / \log\log L, \alpha = d & \text{(Coppersmith, Gamarnik and Sviridenko ('02))} \\ = (\log L)^{\Delta(\alpha) + o(1)}, d < \alpha < 2d & \text{(Biskup (arXiv))} \\ = L^{\theta(\beta) + o(1)} (expected!), & \text{for } p_{\mathbf{xy}} = \beta |\mathbf{x} - \mathbf{y}|^{-2d} \\ \asymp L (expected!), & \text{for } \alpha > 2d. \end{cases}$$

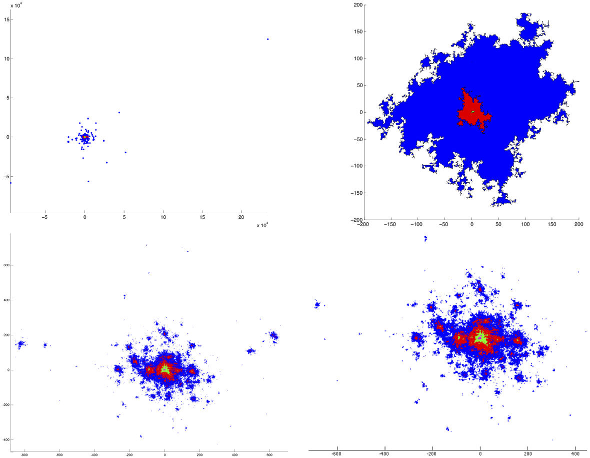


Figure: Growth for $\alpha = 3$ (top two), 3.5 (bottom two)

Simulation for $\mathcal{B}_t^\alpha = \{\mathbf{x} \in \mathbb{Z}^2 : D^\alpha(\mathbf{0}, \mathbf{x}) \le t\}$

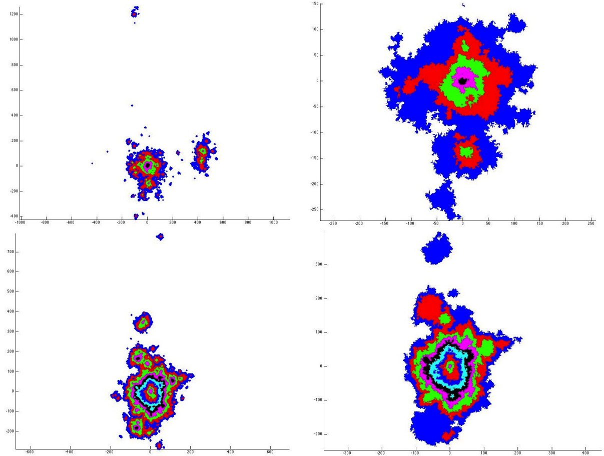


Figure: Growth for $\alpha = 4$ (top two), 4.5 (bottom two)

Simulation for $\mathcal{B}_t^{\alpha} = \{\mathbf{x} \in \mathbb{Z}^2 : D^{\alpha}(\mathbf{0}, \mathbf{x}) \leq t\}$

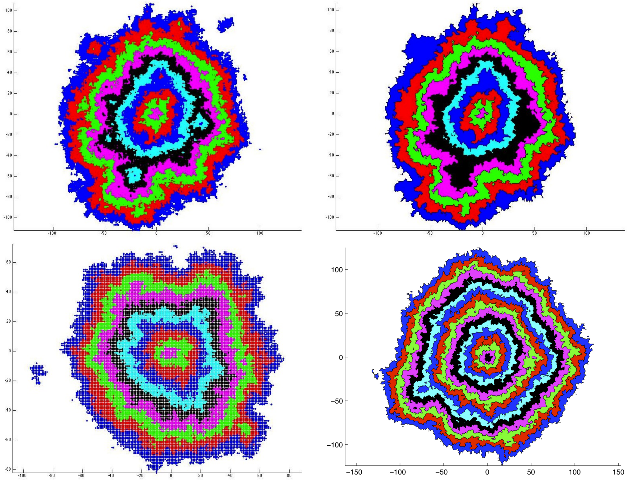
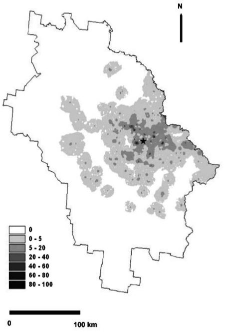


Figure: Growth for $\alpha = 5$ (top two), 5.5 (bottom left), 6 (bottom right)



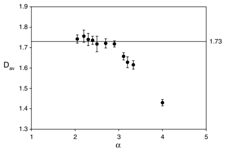


Figure: Abundance of C. grandiflora and box counting plot

Conclusions: Based on the existence of second moment of the dispersal distribution, "$\alpha > 2d$" behavior of the model is same as that of the short range models.

For two dimension, the box counting dimension for the boundary curve is independent of α for $2 < \alpha < 3$ and decreasing for $3 < \alpha < 4$.

Our result: Phase transition

- Instantaneous growth regime: $\alpha < d$.
  Here $\mathcal{B}_t^\alpha = \mathbb{Z}^d$ for any $t > 0$ with probability 1.
- Stretched exponential growth regime: $d < \alpha < 2d$.
  Here, the diameter of $\mathcal{B}_t^\alpha$ is $\exp(t^{1/\Delta + o(1)})$, where $\Delta = \log 2 / \log(2d/\alpha))$ which increases from 1 to ∞ as α goes from d to $2d$.
- Superlinear growth regime: $2d < \alpha < 2d + 1$.
  Here, the diameter of $\mathcal{B}_t^\alpha$ is $t^{1/(\alpha - 2d) + o(1)}$, so the index decreases from ∞ to 1 as α goes from $2d$ to $2d + 1$.
- Linear Growth regime: $\alpha > 2d + 1$.
  Here, the diameter of $\mathcal{B}_t^\alpha$ is $t^{1+o(1)}$.

Remarks:

"Superlinear regime" disproves the first conclusions of CMM ('06).
Phase transitions in the LRP and LRFPP models are not identical.

Instantaneous growth, $\alpha < d$

We show $P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t) = 1$ for any $t > 0$ and $\mathbf{x} \in \mathbb{Z}^d$.

- Fix an integer $K > d/(d-\alpha)$ and let $\ell_j = 2^j(k-1)^j||\mathbf{x}||$.
- Let $B_i^{(j)} := \{\mathbf{y} \in \mathbb{Z}^d : (2i-1)\ell_j \le ||\mathbf{y}|| \le 2i\ell_j\}$ and

$$\mathcal{P}_j := \{\pi = \langle \mathbf{0}\mathbf{x}_1 \ldots \mathbf{x}_{K-1}\mathbf{x} \rangle : \mathbf{x}_i \in B_i^{(j)}\}.$$

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- Using second moment argument

$$P\left(\inf_{\pi\in\mathcal{P}_j} W_\pi^\alpha \leq t\right) \geq \frac{(EN_j)^2}{EN_j^2}, \text{ where } N_j := |\mathcal{P}_j \cap \{\pi : W_\pi^\alpha \leq t\}|.$$

- Using tail bounds for sum of independent exponential random variables we can lower bound $E(N_j)$ and upper bound $E(N_j^2)$

$$P\left(\inf_{\pi\in\mathcal{P}_j} W_\pi^\alpha \leq t\right) \geq \delta > 0 \text{ independent of } j.$$

- So $P(T^\alpha(\mathbf{0},\mathbf{x}) > t) \leq \prod_j P(\inf_{\pi\in\mathcal{P}_j} W_\pi^\alpha > t) = 0$.

Upper bound for $Diam(\mathcal{B}_t^\alpha)$

For $\alpha \in (d, 2d)$, recall $\Delta := 1/\log_2(2d/\alpha) \in (1, \infty)$.
We show $P(T^\alpha(\mathbf{0}, \mathbf{x}) \leq t) \leq \exp\left(c_1 t^{1/\Delta+\varepsilon} - c_2 \ln ||\mathbf{x}||\right)$ for small $\varepsilon > 0$, so for $\alpha \in (d, 2d)$, $T^\alpha(\mathbf{0}, \mathbf{x}) \geq (\ln ||\mathbf{x}||)^{\Delta-\varepsilon}$ w.h.p.

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Sketch: Let $g(u) := E|\mathcal{B}_u^\alpha|$ be the expected volume.
Let $N(\mathbf{x})$ be the number of edges in the optimal path from $\mathbf{0}$ to $\mathbf{x}$.
$N(\mathbf{x})$ small means a long edges has been used in the optimal path:

$$P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t, N(\mathbf{x}) \le a{\cdot}t) \le ||\mathbf{x}||^{-\alpha}(at)^\alpha \int_0^t g(y)[g(t-y)-1]\, dy.$$

$N(\mathbf{x})$ large means too many edges are used within small time.
Using large deviation estimates:

$$P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t, N(\mathbf{x}) > a \cdot t) \le c||\mathbf{x}||^{-\alpha} \exp(-\delta(a)t).$$

Combining (1) and (2) and using $g(t) = \sum_{\mathbf{x}} P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t)$ get a recursive inequality for $g(\cdot)$. Solve it.

Lower bound for diameter

Sketch: Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $2f(x) < x$.
Fix $\mathbf{x}$ with $||\mathbf{x}|| = n$. Define $f_0 = n$ and $f_k = f(f_{k-1})$ inductively.
Min weight of an edge between $B(\mathbf{0}, f_1)$ and $B(\mathbf{x}, f_1)$ is exponential with rate

$$\sum_{\mathbf{u}\in B(\mathbf{0},f_1),\mathbf{v}\in B(\mathbf{x},f_1)} |\mathbf{v}-\mathbf{u}|^{-\alpha} \sim (n-2f_1)^{-\alpha} f_1^{2d}.$$

Let the end points for the minimal edge be $\mathbf{u}_1, \mathbf{v}_1$ respectively.
Consider minimal edge between $B(\mathbf{0}, f_2), B(\mathbf{u}_1, f_2)$ and between $B(\mathbf{v}_1, f_2), B(\mathbf{X}, f_2)$ and proceed similarly.
After k steps, we have 2^k balls of radius f_k and the optimal time is upper bounded by

$$\sum_{i=1}^{k} 2^{i-1}(f_{i-1} - 2f_i)^{\alpha} f_i^{-2d} + 2^k f_k.$$

We have to optimize this over the function f and k.

Lower bound for diameter

- For $d < \alpha < 2d$, we take $f(x) = x^\gamma$ and the optimal $\gamma = \alpha/2d$.
- The optimal k is such that $f_k \approx 1$.
- Here the upper bound for $T^\alpha(\mathbf{0},\mathbf{x})$ shows that $P(T^\alpha(\mathbf{0},\mathbf{x}) \geq (\log ||\mathbf{x}||)^{\Delta+\varepsilon}) \to 0$ fast for any $\varepsilon > 0$.

- For $2d < \alpha < 2d+1$, we take $f(x) = x/a$ with $a > 2$ and the optimal a goes to 2 as $\alpha \uparrow 2d+1$.
- The optimal k is again such that $f_k \approx 1$.
- The upper bound on $T^\alpha(\mathbf{0},\mathbf{x})$ gives $P(T^\alpha(\mathbf{0},\mathbf{x}) \geq ||\mathbf{x}||^{\alpha-2d+\varepsilon}) \to 0$ fast for any $\varepsilon > 0$.

Upper bound for diameter for $2d < \alpha < 2d + 1$

Our previous technique of estimating $P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t)$ gives

$$P(T^\alpha(\mathbf{0}, \mathbf{x}) \le t) \le C(t^\gamma/||\mathbf{x}||)^\alpha \text{ for some } \gamma > 1/(\alpha - 2d),$$

which does not give matching upper bound for diameter.

We show that if $P(Diam(\mathcal{B}_t^\alpha) \le t^\gamma) \to 1$ for some $\gamma > 1/(\alpha - 2d)$, then we can improve γ recursively to have eventually

$$P(Diam(\mathcal{B}_t^\alpha) \le t^{1/(\alpha-2d)+\varepsilon}) \to 1 \text{ for any } \varepsilon > 0.$$

Upper bound for diameter for $2d < \alpha < 2d + 1$

- Suppose $P(Diam(\mathcal{B}_t^\alpha) \le t^\gamma) \to 1$ for some $\gamma > 1/(\alpha - 2d)$.
- W.h.p. no edge of length $\ge t^\delta$ will be used till time t, where $(\gamma d + 1)/(\alpha - d) < \delta$.
- On the above 'good' event, $\mathbf{x} \in \mathcal{B}_t^\alpha$ implies

$$\begin{aligned} t \ge T^\alpha(\mathbf{0}, \mathbf{x}) &\ge \inf_{\mathfrak{p} \in \mathcal{P}_{\mathbf{0},\mathbf{x}}:\ \text{no edge } \ge t^\delta} \sum_{e \in \mathfrak{p}} |e|^\alpha W_e \\ &\ge t^{-\delta(\beta - \alpha)} T^\beta(\mathbf{0}, \mathbf{x}). \end{aligned}$$

- So $\mathcal{B}_t^\alpha \subset \mathcal{B}_{t^{1+\delta(\beta-\alpha)}}^\beta$ w.h.p.
- If β just crosses $2d + 1$, then assuming linear growth of $T^\beta(\mathbf{0}, \mathbf{x})$ w.h.p. the diameter of $\mathcal{B}_t^\alpha$ is at most $1 + \delta(\beta - \alpha) < \gamma$.

Linear growth for $\alpha > 2d + 1$

- Nearest neighbor growth ensures at most linear growth for $T^{\alpha}(\mathbf{0}, \mathbf{x})$.
- In view of subadditivity, It suffices to show at least linear growth for $T^{\alpha}(\mathbf{0}, \mathbf{x})$.
- The key is that if
$$\theta \in (\frac{d+1}{\alpha - d}, 1),$$
then no edge of length n^{θ} is used within time n by a vertex in the Euclidean ball of diameter n.
- Divide $[0, n]^d$ into a grid of size n^{θ}, and the optimal path from $\mathbf{0}$ to $n\mathbf{e}_1$ must jump from a box to a nearest neighbor box.
- It crosses $O(n^{1-\theta})$ such boxes and takes at least $O(n^{\theta})$ time for a fraction of them by a renormalization argument.

Future questions

- Study the critical values $\alpha = d, 2d, 2d + 1$.
- Properties of rescaled growth set.
- Bounds for boundary fluctuations.
- On a torus study the time evolution of the fraction covered.

Thank You